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Mass equivalence of four-bar linkages for the design of reconfigurable force-balanced mechanisms

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Abstract. In this paper the mass equivalence of four-bar linkages is investigated, which is the characteristic that the mass of the complete linkage can be modeled dynamically equivalent with one or more equivalent masses in one or more points in the linkage. For a general spatial or planar four-bar linkage this results in a model with two equivalent masses, one in a point in each of two opposite links. Since a four-bar consists of two sets of opposite links, its mass can be modeled with two different mass equivalent models and therefore there exists a mass equivalent point in each of the four links. It is shown how these four mass equivalent points can be used for designing a balanced linkage with a floating four-bar linkage that can be reconfigured into four different compositions, depending on which equivalent points are applied as joints to connecting links. It is also shown that when the four-bar linkage becomes a parallelogram, the four mass equivalent points become mass equivalent lines, allowing smooth reconfiguration to numerous positions along the links.

Keywords: mass equivalent model, four-bar linkage, force balance, parallelogram, reconfigurable

1 Introduction

Modeling the mass of mechanism elements with equivalent masses has shown to be a fruitful approach in statics and dynamics for long time. In robotics it is commonly used for deriving the balance conditions of closed loop mechanisms, modeling the mass of the moving platform or of one link in each closed loop in order to obtain open-loop chains [11, 2] or for finding optimally balanced solutions [3].

Mass equivalent modeling has also shown fruitful in the synthesis of (complex) balanced linkages where a single mass model can represent a variety of multi-degree-of-freedom (multi-DoF) mass equivalent linkages, which then can be exchanged with one another without affecting the force balance. Examples have been shown of mass equivalent dyads [5], mass equivalent triads [6], and mass equivalent pantographs [10], which are also applied for constructing balanced focal linkages [8] and for balancing the Peaucellier-Lipkin straight-line linkage [9]. This synthesis approach also allows a relatively simple way to derive the force balance conditions of complex linkages [7].

This paper is focussed on the mass equivalence of a general (spatial or planar) four-bar linkage of which the mass can be modeled with a single mass model. Without paying notice to it, Dobrovolskii [4] already made use of the mass equivalent points of the planar 4R four-bar linkage by applying a pantograph to these points to trace the center of mass of the linkage. In this paper the mass equivalence of the four-bar linkage is investigated in dept. First the mass equivalent points of the general four-bar linkage are obtained and subsequently they are applied for the design of a reconfigurable mechanism with a floating four-bar linkage that can be reconfigured in four different compositions without affecting the force balance. The last part presents the specific situation that the general four-bar linkage has become a planar parallelogram in which mass equivalent lines allow smooth reconfiguration to a wide variety of poses.

2 Mass Equivalent Points for Force Balance

In Fig. 1 a force-balanced linkage of 7 elements is presented. It consists of a floating four-bar linkage $B_0B_1B_2B_3$ and links A_0A_1 and A_2A_3 of which A_0 and A_3 are pivots with the base link A_0A_3 and A_1 and A_3 are connected with a joint in points R_1 and R_3, respectively. R_1 and R_3 are points in links B_0B_1 and B_2B_3, respectively, located at a distance r_1 and r_3, respectively, from the joint as illustrated and are mass equivalent points of the four-bar, as will become clear later. This linkage can also be observed as a four-bar linkage $A_0A_1A_2A_3$ of which

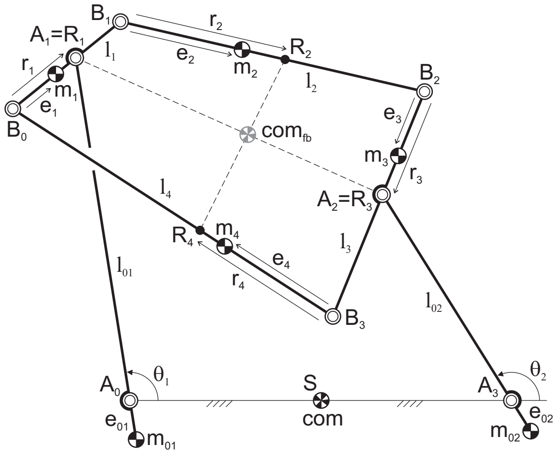

Fig. 1. Force-balanced reconfigurable linkage with common CoM in S for any pose, based on a floating four-bar $B_0B_1B_2B_3$ that is connected to the base with two links.

the coupler link has been replaced with the floating four-bar linkage $B_0B_1B_2B_3$. All the joints can be considered spherical joints, allowing spatial motions in all directions, but they can also be considered as universal or revolute joints in any desired combination. When all the joints would be revolute pairs with their axes of rotation parallel, then Fig. 1 would show a 2-DoF planar linkage with the DoFs θ_1 and θ_2 as illustrated.

Each link i of the floating four-bar has a length l_i and a mass m_i of which the link center-of-mass (CoM) is located at a distance e_i from the link joint as illustrated. Links A_0A_1 and A_2A_3 have a length l_{01} and l_{02}, respectively, and a mass m_{01} and m_{02}, respectively, located at a distance e_{01} and e_{02}, respectively, from the base joints as illustrated. These link masses act also as countermasses in order to balance the complete linkage such that the common CoM of all the links is in a fixed point S in the base for any pose or motion of the linkage.

To calculate the conditions for which the common CoM of all the links is in S, the floating four-bar linkage can be modeled mass equivalently as shown in Fig. 2. The mass of link B_1B_2 is modeled with an equivalent mass $m_2^a = m_2(1 - \frac{e_2}{l_2})$ in B_1 and $m_2^b = m_2\frac{e_2}{l_2}$ in B_2 while the mass of link B_3B_0 is modeled similarly with equivalent masses $m_4^a = m_4(1 - \frac{e_4}{l_4})$ in B_3 and $m_4^b = m_4\frac{e_4}{l_4}$ in B_0. With these equivalent masses the links B_1B_2 and B_3B_0 can be eliminated, resulting in two independend parts of the linkage.

The common CoM of m_1, m_2^a, and m_4^b in link B_0B_1 is defined as point R_1 while the common CoM of m_3, m_2^b, and m_4^a in link B_2B_3 is defined as point R_3 of which their locations are calculated as:

$$r_1 = \frac{m_1e_1 + m_2^al_1}{m_1 + m_2^a + m_4^b}, \quad r_3 = \frac{m_3e_3 + m_4^al_3}{m_3 + m_4^a + m_2^b} \tag{1}$$

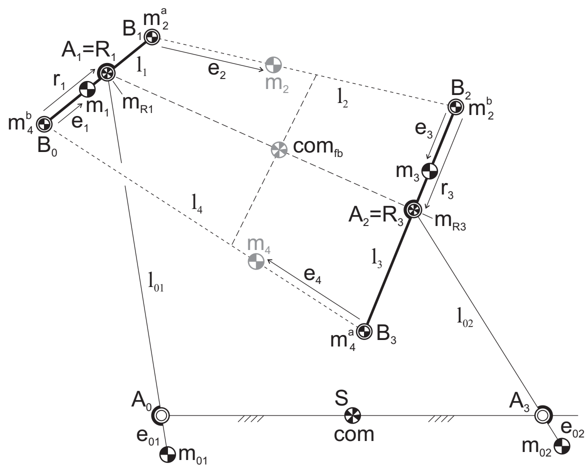

Fig. 2. Equivalent mass model of the floating four-bar linkage with equivalent masses m_{R1} and m_{R3} in mass equivalent points R_1 and R_3, respectively.

R_1 and R_3 are mass equivalent points of the floating four-bar, which means that the motion of the floating four-bar mass $m_{fb} = m_1 + m_2 + m_3 + m_4$ can be modeled mass equivalently with an equivalent mass $m_{R_1} = m_1 + m_2^a + m_4^b$ in point R_1 and an equivalent mass $m_{R_3} = m_3 + m_4^a + m_2^b$ in point R_3. With these equivalent masses, the two separate parts of the model in Fig. 2 must be balanced individually. Therefore A_1 must be located in R_1 and A_2 must be located in R_3 to have the floating four-bar balanced about A_1 and A_2. Subsequently the CoMs of links A_0A_1 and A_2A_3 need to fulfill the balance conditions:

$$e_{01} = \frac{(m_1 + m_2^a + m_4^b)l_{01}}{m_{01}}, \quad e_{02} = \frac{(m_3 + m_4^a + m_2^b)l_{02}}{m_{02}} \tag{2}$$

with which the common CoM of the left part of the linkage model in Fig. 2 is in A_0 and the common CoM of the right part of the linkage model is in A_3 for any pose and motion. Since A_0 and A_3 are base pivots, the common CoM of the complete linkage is in a stationary point in the base as well, which is in point S.

For the balance conditions (1) and (2) the linkage in Fig. 1 is force balanced for any possible variation. Figure 3 shows, for instance, a variation in which the floating four-bar linkage is inverted and the base joints A_0 and A_3 are in a single point. This linkage is force balanced with the common CoM in this single base point for any pose.

It is possible to follow the same procedure to find the mass equivalent points R_2 and R_4 of the floating four-bar, which are located in links B_1B_2 and B_3B_0, respectively, at distances r_2 and r_4 from the link joint, respectively, as illustrated in Fig. 1. In this case the mass of link B_0B_1 is modeled with equivalent masses $m_1^a = m_1(1 - \frac{e_1}{l_1})$ and $m_1^b = m_1\frac{e_1}{l_1}$ located in B_0 and B_1, respectively, and the mass of link B_2B_3 is modeled with equivalent masses $m_3^a = m_3(1 - \frac{e_3}{l_3})$ and $m_3^b = m_3\frac{e_3}{l_3}$ located in B_2 and B_3, respectively. The location of points R_2 and

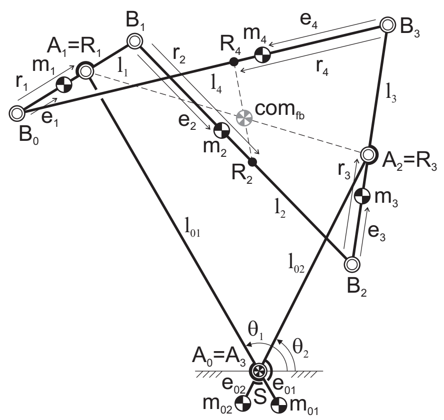

Fig. 3. Force-balanced linkage with inverted floating four-bar and the base pivots in a single point, a variation of the design in Fig. 1.

R_4 in their link then is found with:

$$r_2 = \frac{m_2 e_2 + m_3^a l_2}{m_2 + m_3^a + m_1^b}, \quad r_4 = \frac{m_4 e_4 + m_1^a l_4}{m_4 + m_1^a + m_3^b} \tag{3}$$

It is interesting to note that the line through R_1 and R_3 and the line through R_2 and R_4 intersect at the common CoM of the floating four-bar for any pose. It is also interesting that for force balance the floating four-bar could be mounted on links A_0A_1 and A_2A_3 in four different ways, for instance as well by placing joint A_1 in R_4 and joint A_2 in R_2 as illustrated in Fig. 4c. This results in a model with an equivalent mass $m_{R_4} = m_4 + m_1^a + m_3^b$ located in A_1 and an equivalent mass $m_{R_2} = m_2 + m_3^a + m_1^b$ located in A_2. For force balance the CoMs of links A_0A_1 and A_2A_3 then need to fulfill the balance conditions:

$$e_{01} = \frac{(m_4 + m_1^a + m_3^b)l_{01}}{m_{01}}, \quad e_{02} = \frac{(m_2 + m_3^a + m_1^b)l_{02}}{m_{02}} \tag{4}$$

3 Reconfiguration to Four Force-balanced Compositions

With the four mass equivalent points, it is possible to mount the floating four-bar in four different ways on the supporting links A_0A_1 and A_2A_3, as illustrated in Fig. 4. However, since the equivalent masses of the floating four-bar m_{R_1}, m_{R_2}, m_{R_3}, and m_{R_4} are not equal in general, this has influence on the design of the supporting links which then also have different balance conditions for each of the four configurations, which already becomes clear by comparing Eqs. (2) and (4) of the configurations in Fig. 4a and 4c. In this section three different cases are presented for the floating four-bar to be reconfigured from one composition to another without affecting the balanced design by dismounting and reassembling it onto A_1 and A_2.

3.1 Case 1, one reconfiguration possibility of R_1 to R_4 and R_3 to R_2

To reconfigure the linkage from Fig. 4a to Fig. 4c such that force balance is maintained implies that the balance conditions for the supporting links, Eqs. (2) and (4), must be equal. Since this reconfiguration consists of mounting joints A_1 and A_2 in either R_1 and R_3 or R_4 and R_2, respectively, the additional balance condition for which force balance is maintained follows also from the equivalent four-bar masses that need to be equal with $m_{R_1} = m_{R_4}$ and $m_{R_3} = m_{R_2}$, which can be written as:

$$\begin{aligned} m_1 + m_2^a + m_4^b &= m_4 + m_1^a + m_3^b \\ m_3 + m_4^a + m_2^b &= m_2 + m_3^a + m_1^b \end{aligned} \tag{5}$$

and which results in the single force balance condition:

$$m_1^b + m_2^a - m_3^b - m_4^a = 0 \tag{6}$$

or after substitution:

$$m_1 \frac{e_1}{l_1} + m_2(1 - \frac{e_2}{l_2}) - m_3 \frac{e_3}{l_3} - m_4(1 - \frac{e_4}{l_4}) = 0 \tag{7}$$

3.2 Case 2, one reconfiguration possibility of R_1 to R_2 and R_3 to R_4

Similarly as for case 1, it is possible to reconfigure the linkage from Fig. 4a to Fig. 4d such that force balance is maintained without modifying the supporting links. This reconfiguration consists of mounting joints A_1 and A_2 in either R_1 and R_3 or R_2 and R_4, respectively. The additional balance condition for which force balance is maintained follows from the equivalent four-bar masses that need to be equal with $m_{R_1} = m_{R_2}$ and $m_{R_3} = m_{R_4}$, which can be written as:

$$
\begin{aligned}
m_1 + m_2^a + m_4^b &= m_2 + m_3^a + m_1^b \\
m_3 + m_4^a + m_2^b &= m_4 + m_1^a + m_3^b
\end{aligned}
\tag{8}
$$

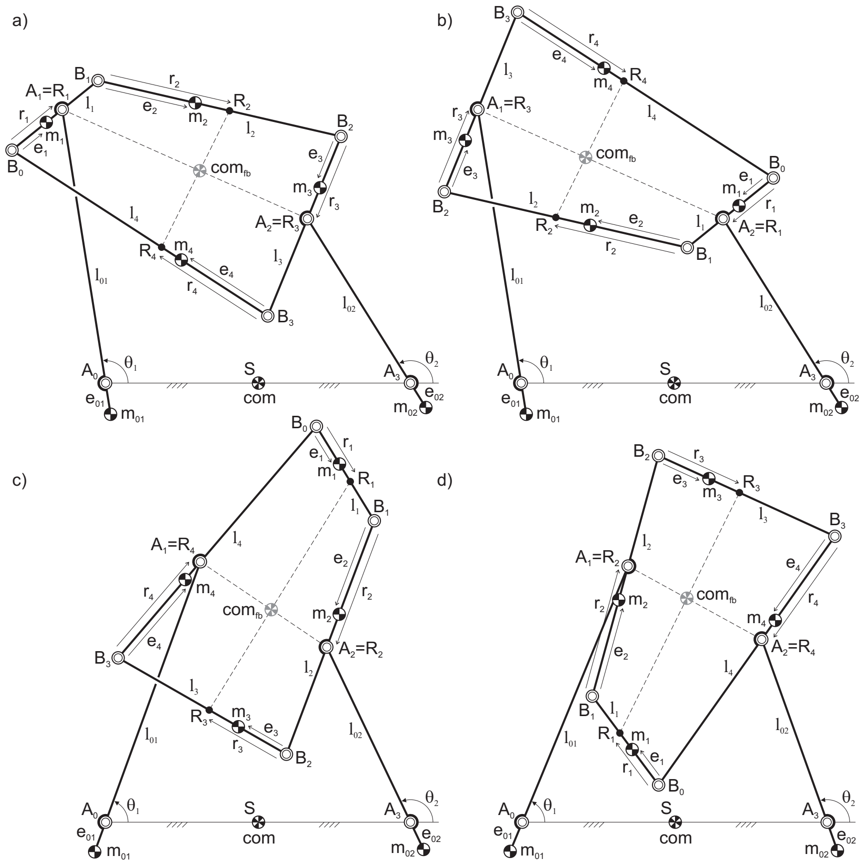

Fig. 4. Four different ways of mounting the floating four-bar on the supporting links A_0A_1 and A_2A_3 for balance. Reconfiguration of the linkage from one composition to another is possible without affecting the force balance.

and which results in the single force balance condition:

$$m_1^a - m_2^b - m_3^a + m_4^b = 0 \tag{9}$$

or after substitution:

$$m_1(1 - \frac{e_1}{l_1}) - m_2\frac{e_2}{l_2} - m_3(1 - \frac{e_3}{l_3}) + m_4\frac{e_4}{l_4} = 0 \tag{10}$$

3.3 Case 3, independent reconfiguration to any R_i

To reconfigure the floating four-bar linkage from any composition to any other composition in Fig. 4 while maintaining force balance it is required that all equivalent masses of the floating four-bar are equal, i.e. $m_{R_1} = m_{R_2} = m_{R_3} = m_{R_4}$. From this it is derived that both the balance condition of case 1, Eq. (7), and of case 2, Eq. (10), need to hold at the same time. It has been numerically verified that these conditions result in realistic, natural, and practical designs.

4 Reconfigurable Parallelogram Linkage

It is interesting to discover the specialities of the linkage when the floating four-bar linkage is constructed as a planar parallelogram with $l_1 = l_3$ and $l_2 = l_4$ as shown in Fig. 5. Instead of the ability of all the joints being of any type - spherical, universal, or revolute - as in Fig. 1, here the joints in B_0, B_1, B_2, and B_3 must have revolute pairs to obtain a planar parallelogram $B_0B_1B_2B_3$. The

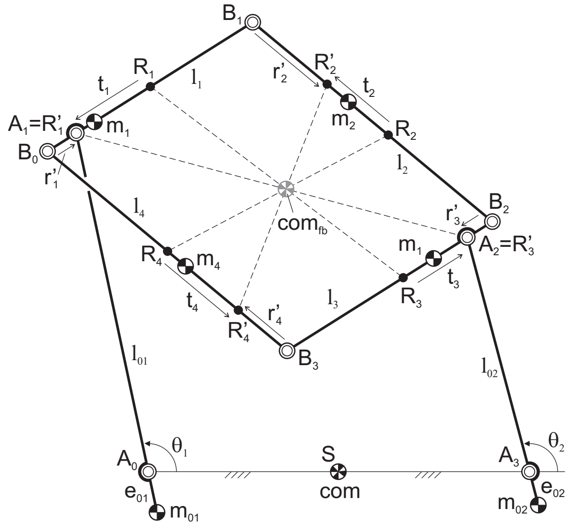

Fig. 5. Force-balanced reconfigurable linkage with CoM in S for any pose, based on a floating parallelogram connected to the base with supporting links A_0A_1 and A_2A_3.

other joints A_0, A_1, A_2, and A_3 can still be of any type such that the planar parallelogram can be moved spatially.

Since the parallelogram is known for its properties of similarity [1], caused by opposite links moving synchronously and referred to as a pantograph, it can be modeled mass equivalently not solely in a single mass equivalent point in each link as for the general four-bar linkage, but in infinitely many points along each link. These points R'_i are located at distances r'_i from the indicated link joint and can be described with distances t_i from the initial mass equivalent points R_i as illustrated. With r_1 and r_3 calculated as in Eq. 1, the locations of R'_1 and R'_3 can be described with:

$$r'_1 = r_1 - t_1 = \frac{m_1 e_1 + m_2^a l_1}{m_1 + m_2^a + m_4^b} - t_1 \tag{11}$$
$$r'_3 = r_3 - t_3 = \frac{m_3 e_3 + m_4^a l_3}{m_3 + m_4^a + m_2^b} - t_1 \frac{m_1 + m_2^a + m_4^b}{m_3 + m_4^a + m_2^b}$$

in which the relation between t_1 and t_3 is determined with $m_{R_1} t_1 = m_{R_3} t_3$ such that the line through R'_1 and R'_3 intersects with the CoM of the parallelogram. With r_2 and r_4 calculated as in Eq. 3, the positions of R'_2 and R'_4 can be found similarly with:

$$r'_2 = r_2 - t_2 = \frac{m_2 e_2 + m_3^a l_2}{m_2 + m_3^a + m_1^b} - t_2 \tag{12}$$
$$r'_4 = r_4 - t_4 = \frac{m_4 e_4 + m_1^a l_4}{m_4 + m_1^a + m_3^b} - t_2 \frac{m_2 + m_3^a + m_1^b}{m_4 + m_1^a + m_3^b}$$

in which the relation between t_2 and t_4 is determined with $m_{R_2} t_2 = m_{R_4} t_4$. Instead of defining the mass equivalent points, these equations can be said to define the mass equivalent lines of the parallelogram linkage.

By mounting joints A_1 and A_2 to R'_1 and R'_3, respectively, and having these points slide along the links following conditions (11), the reconfigurable force balanced linkage in Fig. 6 is obtained. In fact there can be real slider elements in R'_1 and R'_3 of which the mass can be included in m_{R_1} and m_{R_3} for balance. The linkage is force balanced not only for any pose and motion of the linkage with R'_1 and R'_3 fixed in their link, but also for any pose and motion of R'_1 and R'_3 along their link, gaining an additional DoF.

While in Fig. 6 the two slider elements are not mechanically constrained to move synchronously according the conditions of Eq. (11), in Fig. 7 a solution for this is presented with two additional links which intersect in the CoM of the parallelogram where they have a common revolute pair. One link is parallel to links B_0B_1 and B_2B_3 and has joints with links B_1B_2 and B_3B_0 while the other link is connected with sliders to the revolute joints in R'_1 and R'_3 and hence maintains the condition between the two points. Also the mass of these additional elements can be included for force balance of the complete linkage. It is interesting to note that for $r'_1 + r'_3 = l_1$, leading to $t_1 = \frac{(r_1+r_3-l_1)m_{R_3}}{m_{R_1}+m_{R_3}}$, in Eqs. (11) the line through R'_1 and R'_3 is parallel to links B_1B_2 and B_3B_0.

Also for the floating parallelogram the four different ways for reconfiguration as shown in Fig. 4 are possible, following the same conditions as presented for each case in the previous section. Altogether this leads to a wide variety of possible reconfigurable designs for which force balance is maintained. In the

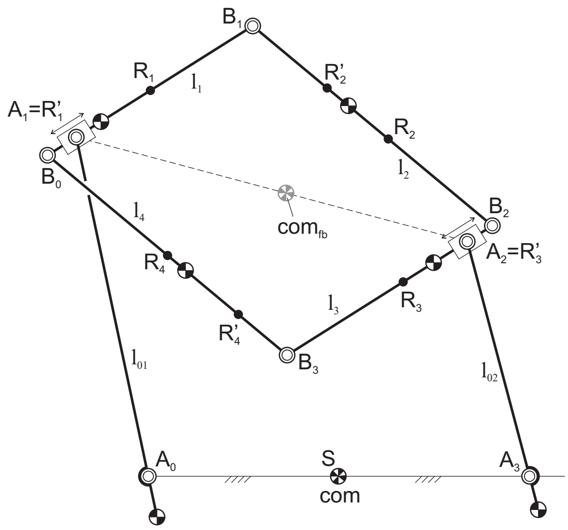

Fig. 6. Force-balanced reconfigurable linkage solution with sliders in R_1' and R_3'.

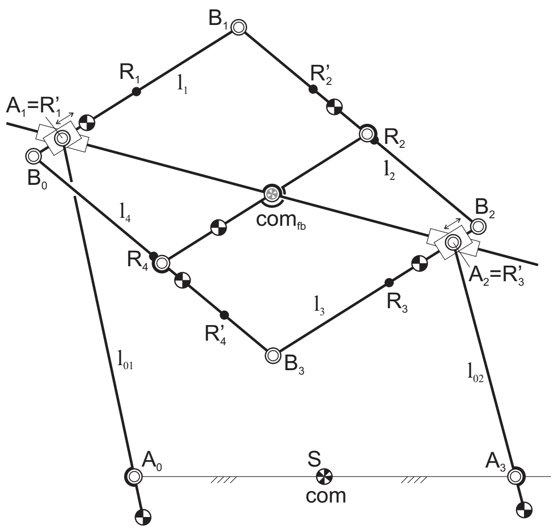

Fig. 7. Force-balanced reconfigurable linkage solution where the sliders in R_1' and R_3' are mechanically constrained by additional elements to move proportionally.

solution of Fig. 7 the link through R_1' and R_3' could also be considered as a kind of coupler link of the four-bar linkage $A_0A_1A_2A_3$ mounted with sliders in A_1 and A_2. Hence the linkage in Fig. 7 could be regarded a way to balance such four-bar linkage with a sliding coupler link. Finding other and even better means to synchronize the motions of the two sliders in R_1' and R_3' may be an interesting target for further studies, e.g. investigating solutions with cables and pulleys along the four-bar links.

5 Conclusions

In this paper the mass equivalent models of a general four-bar linkage were presented. It was shown that there is a mass equivalent point in each of the four links and that these points can be applied for the design of force-balanced linkages that can be reconfigured into four different compositions without affecting the balance. It was also shown that when the general four-bar linkage is reduced to a planar parallelogram, the mass equivalent points become mass equivalent lines, allowing smooth reconfiguration to a wide variety of configurations without affecting the force balance.

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